



NAVAL POSTGRADUATE SCHOOL

MONTEREY, CALIFORNIA

THESIS

**SOLVING FOR OPTIMAL RETIREMENT FINANCIAL
PLANS BY MAXIMIZING A DISCOUNTED HABIT
FORMATION UTILITY FUNCTION**

by

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March 2009

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**SOLVING FOR OPTIMAL RETIREMENT FINANCIAL PLANS BY
MAXIMIZING A DISCOUNTED HABIT FORMATION UTILITY FUNCTION**

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ABSTRACT

With the increasing popularity of defined-benefit retirement plans, retiring individuals are looking for professional financial advice to help manage their nest eggs. Commonly prescribed generic “one size fits all” rules of thumb such as the 4% Rule can carry someone successfully through retirement, but they do not effectively take into account individual expectations or preferences of the retiree or the volatility and risk of the capital markets. An alternative approach is an investment strategy focused on maximizing an individual’s utility or “happiness” during retirement. We consider the maximization of a utility function that exhibits habit formation.

The programming language C++ is used to implement a solution algorithm for this maximization problem, and Microsoft Excel is utilized as an interface to present and analyze data. The resulting implementation is a planning tool that provides optimized retirement financial plans according to an individual’s preferences.

We demonstrate the effects of habit formation on optimal retirement consumption and investment plans and show how a dynamic investment and spending strategy that maximizes an individual’s utility can provide a major improvement over the rules of thumb currently practiced.

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EXECUTIVE SUMMARY

With the increasing popularity of defined-benefit retirement plans, retiring individuals are now finding themselves in charge of developing investment and spending strategies that allow them to comfortably live through their golden years without running out of money. Expectedly, many retirees are looking for professional financial advice to help manage their nest eggs. Everyone has different needs and expectations for their retirement, but the professional advice commonly prescribed is the use of generic “one size fits all” rules of thumb. One of the most common rules of thumb is the 4% Rule, which recommends annually spending a fixed real amount equal to 4% of initial wealth and then adjusting portfolio composition to maintain constant proportions of equities and bonds. While rules of thumb can carry someone successfully through retirement, they do not effectively take into account individual expectations or preferences of the retiree or the volatility and risk of the capital markets.

An alternative to the rule of thumb approach is an investment strategy focused on maximizing an individual’s utility or “happiness” during retirement. This thesis defines, solves and analyzes a utility maximization problem that includes personal preference factors for a given retirement period. Notably, we consider a utility function that exhibits habit formation, i.e., happiness derived from consumption in the future depends on a retiree’s prior history of consumption.

The programming language C++ is used to implement a solution algorithm and Microsoft Excel is utilized as an interface to present and analyze data. The resulting implementation is a planning tool that provides optimized retirement financial plans according to an individual’s preferences.

We demonstrate the effects of habit formation on optimal retirement consumption and investment plans and show how a dynamic investment and spending strategy that maximizes an individual’s utility can provide a major improvement over the rules of thumb currently practiced.

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I. INTRODUCTION

A. BACKGROUND

Over the past few decades, the bulk of U.S. private-sector retirement assets have shifted from traditional defined-benefit (pension) plans to defined-contribution (401(k)-type) plans (MacDonald 2006). Under a defined-benefit retirement plan, it is the responsibility of the employer to manage the pension assets and provide its beneficiaries with consistent, secure income throughout retirement. Under a defined-contribution plan, these responsibilities become that of the retiree. Retiring individuals are now finding themselves in charge of developing investment and spending strategies that allow them to comfortably live through their golden years without running out of money. Expectedly, many retirees are looking for professional financial advice to help manage their nest eggs.

Everyone has different needs and expectations for their retirement. Unfortunately, the professional advice commonly offered is not a personalized strategy of investment and spending tailored to each individual. Instead, retirees are most often prescribed generic “one size fits all” rules of thumb. One of the most common rules of thumb is the 4% Rule, which recommends annually spending a fixed real amount equal to 4% of initial wealth and then adjusting portfolio composition to maintain constant proportions of equities and bonds. Another popular investment rule is a glide-path strategy that annually increases the bond portion of a portfolio. While these rules can carry someone successfully through retirement, they do not effectively take into account individual expectations or preferences of the retiree or the volatility and risk of the capital markets.

An alternative to the rule of thumb approach is an investment strategy focused on maximizing an individual’s utility or “happiness” during retirement. Sharpe, Scott and Watson (2008) suggest a dynamic investment and spending strategy that maximizes an individual’s utility can provide a major improvement over the rules of thumb currently practiced.

B. RESEARCH GOAL

This thesis defines, solves and analyzes a utility maximization problem that includes personal preference factors for a given retirement period. Notably, we consider a utility function that exhibits habit formation, i.e., happiness derived from consumption in the future depends on a retiree's prior history of consumption. The programming language C++ is used to implement a solution algorithm and Microsoft Excel is utilized as an interface to present and analyze data. The resulting implementation is a planning tool that provides optimized retirement financial plans according to an individual's preferences.

C. ASSUMPTIONS

1. Retiree

The retiree's preferences must be adequately represented by the habit forming utility function, as defined precisely in Chapter II. The retiree must believe that more consumption is better than less consumption, and that consumption today is better than consumption tomorrow. The retiree must also wish to consume in such a way that maximizes his/her utility.

2. Capital Market

We restrict our analysis to a model economy that follows an annual binomial model. The market consists of a risk-free asset (bonds) and a volatile asset (stocks). The volatile asset is assumed to go either 'up' or 'down' each year with equal probability. As further explained in Chapter II, markets are complete.

3. The Retirement Scenario

The retiree begins with a certain amount of wealth that is to support him/her throughout a retirement of 30 years. All of this initial wealth will be immediately invested into the two-asset market mentioned above and described in detail later. At the beginning of each year the retiree will withdraw a lump sum of money to be consumed completely before the next year when it is time to withdraw again. Remaining wealth must equal zero at the retirement horizon. We refer to one year as a time period.

Figure 1 is a graphical ‘tree’ representation of our scenario. Each node represents a retiree’s consumption $c_{t,s}$ at time t and in market path s . The arcs connecting the consumption nodes represent the transition from one node to another based on the outcome of the market, i.e., “up arc” if an up market is experienced or a “down arc” when down markets occur. Market path s is determined by the specific sequence of up and down markets experienced along a particular path of consumption nodes. For example, consumption at time $t = 2$, market path $s = 1$ represents an up market at time 0 and an up market at time 1. At $t = 2$, market path $s = 3$ represents a down market at time 0 and then an up market at time 1. As time advances, the number of possible market paths S_t in time period t grows exponentially. The retiree’s consumption during the last year of employment $t = -1$ is $c_{-1,1}$ and is assumed known. We consider the retiree’s consumption now ($t = 0$), and in the next 30 years ($t = 1, 2, \dots, 30$).

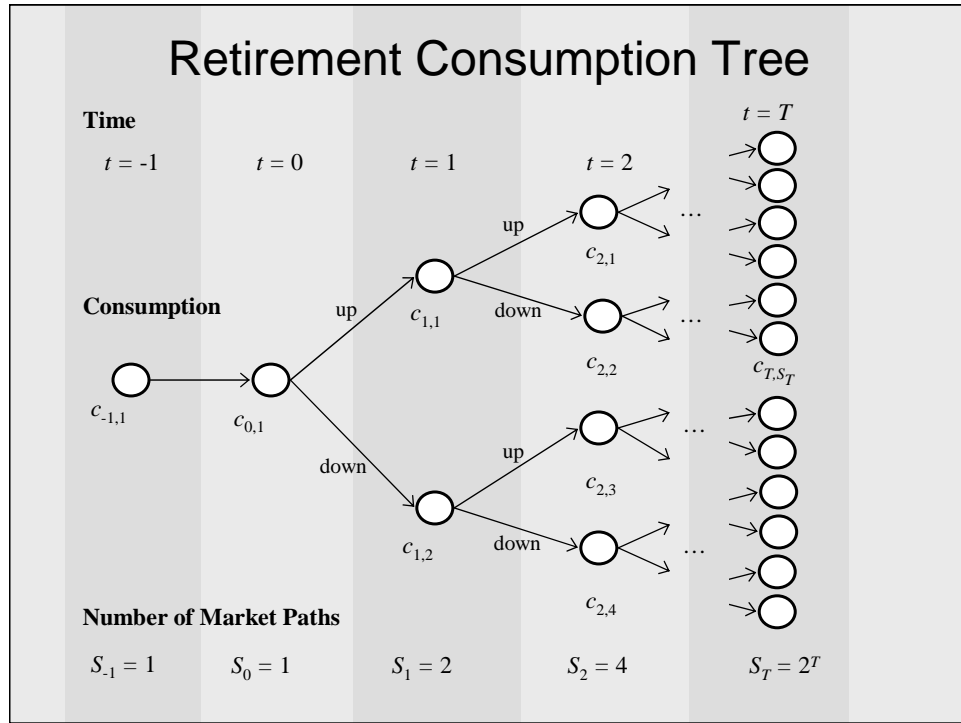


Figure 1. Retirement Consumption Tree

D. LITERATURE REVIEW

Sharpe, Scott and Watson (2008) state that an individual with a defined-contribution plan must develop a retirement financial strategy composed of both an investment and a spending (consumption) strategy. Their retirement financial strategy can only be efficient when both investment and spending decisions are made together as part of a complete retirement financial strategy. They analyze two rules of thumb, the 4% spending rule and a glide slope strategy, and determine them to be inefficient financial strategies because they attempt to support a constant consumption rate by investing in a volatile capital market. Furthermore, in Scott, Sharpe, and Watson (2009) it is demonstrated that financial rules of thumb can actually be quite wasteful of an individual's retirement savings by gathering surpluses when market prices increase and deficits when market prices decrease.

Goldstein, Johnson, and Sharpe (2008) state that for a defined-contribution plan there is a probability distribution of wealth associated with market performance and investment strategy. Many investors do not know or do not like their probability distributions given by their current investments. Therefore, many investors' portfolios are not consistent with their preferences and are not going to provide them with the retirement they desire. A tool is developed to allow investors to estimate their individual preferences by considering multiple wealth distribution outcomes. With this knowledge an investor can make better informed investment decisions.

Gomes and Michaelides (2003) suggest that consumption in the previous period $t - 1$ quite possibly affects the amount of utility experienced by consumption in period t for a retiree. If this is true, a time separable utility function does not accurately measure the "goodness" of a retirement consumption plan. For these individuals, a habit forming utility function is a better suited tool for developing and evaluating retirement consumption and investment plans.

Sharpe (2007a) suggests that quite often investors take the traditional approach of focusing exclusively on mean (expected return) and variance when evaluating portfolio returns. He conducts an alternative analysis using expected utility to evaluate the

efficiency of the mean-variance approach by estimating possible future returns and investor's preferences. It is shown that the expected utility approach under certain assumptions can provide the same results as the mean/variance approach.

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II. MODEL AND ALGORITHM DEVELOPMENT

A. UTILITY

1. Discounted Time Separable Utility

The discounted time separable utility function $\hat{U}_{t,s}(c_{t,s})$ is defined in terms that affect a retiree's consumption and investment decisions (Goldstein, Johnson, and Sharpe 2008). These parameters are denoted as follows:

$c_{t,s}$ consumption at time t in market path s , with $t = -1, 0, \dots, T$,
and $s = 1, \dots, 2^T$ if $t \geq 0$ and $s = 1$ if $t = -1$.

a_t time discount factor at time t (represents investor's time preference
relative to consumption sooner rather than later).

g_t risk aversion coefficient for time t (represents the investor's
propensity to accept risk).

The discounted time separable utility function for time t and market path s is defined as:

$$\hat{U}_{t,s}(c_{t,s}) = \frac{a_t (c_{t,s})^{1-g_t}}{1-g_t} \quad (1)$$

This function satisfies $\frac{\partial \hat{U}_{t,s}}{\partial c_{t,s}}(c_{t,s}) > 0$ and $\frac{\partial^2 \hat{U}_{t,s}}{\partial c_{t,s}^2}(c_{t,s}) < 0$. Each period's utility is independent and can be calculated without knowledge of previous or future consumption. A retirement consumption plan's overall expected utility can be calculated simply by summing the utility experienced over all times and market paths weighted by the market path probability. As illustrated in Figure 2, this nonlinear function is both continuous and concave. Increases in consumption have increasing, but diminishing, returns in utility.

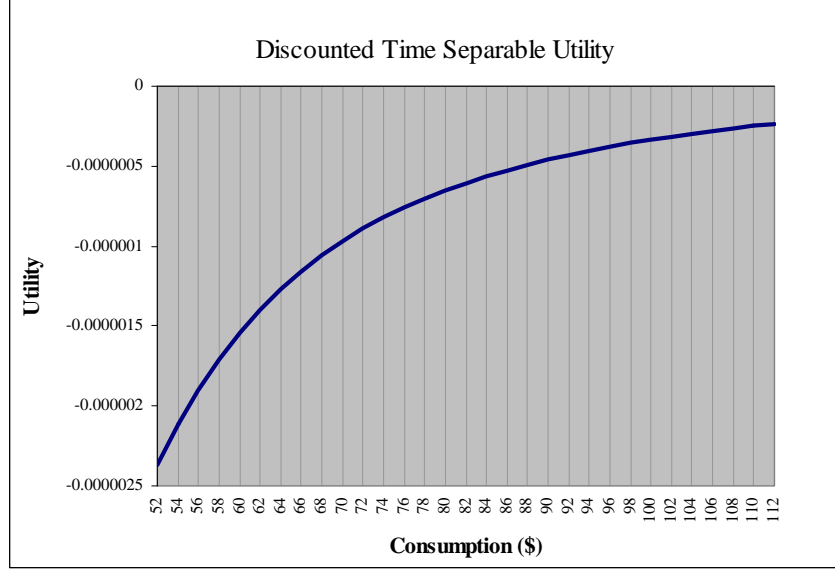


Figure 2. Discounted Time Separable Utility ($g_t = 4$)

2. Discounted Habit Formation Utility

Watson (2008) introduces a habit formation utility function that includes the same terms as the time separable utility and adds a habit formation parameter d_t . Maximizing this function and analyzing the effects habit formation has on spending preferences will be a focus of the next chapter.

d_t habit formation coefficient at period t (represents the propensity of the investor to value the consumption at time t relative to consumption at time $t-1$).

The habit forming utility function is defined by:

$$U_{t,s}(c_{t,s}, c_{t-1, \lceil s/2 \rceil}) = \frac{a_t (c_{t,s} - d_t c_{t-1, \lceil s/2 \rceil})^{1-g_t}}{1-g_t} \quad (2)$$

where $\lceil s/2 \rceil$ denotes the smallest integer at least as large as $s/2$. We observe that $c_{t-1, \lceil s/2 \rceil}$ is the consumption in the preceding time period and market path. Note that utility at time t and market path s depends only on $c_{t,s}$ and $c_{t-1, \lceil s/2 \rceil}$.

When $d_t = 0$, (2) becomes identical to the time separable utility. Assuming $d_t > 0$, this utility function is not defined for the condition $c_{t,s} \leq d_t c_{t-1, \lceil s/2 \rceil}$. As d_t increases, the more important increases in consumption from $t - 1$ to t become to keep utility positive. This function is continuous and concave for both consumption variables.

B. MODEL MARKET

As mentioned in Chapter I, we restrict our analysis to a model economy that follows an annual binomial model. We use the same simple complete market as Sharpe, Scott, and Watson (2008), Watson (2008), and Sharpe (2007b). The market consists of a risk-free asset (bonds) and a volatile asset (stocks). The volatile asset is assumed to go either ‘up’ or ‘down’ each year with equal probability. An example of a three period retirement ($t = 0, 1, 2$) scenario as shown in Figure 3.

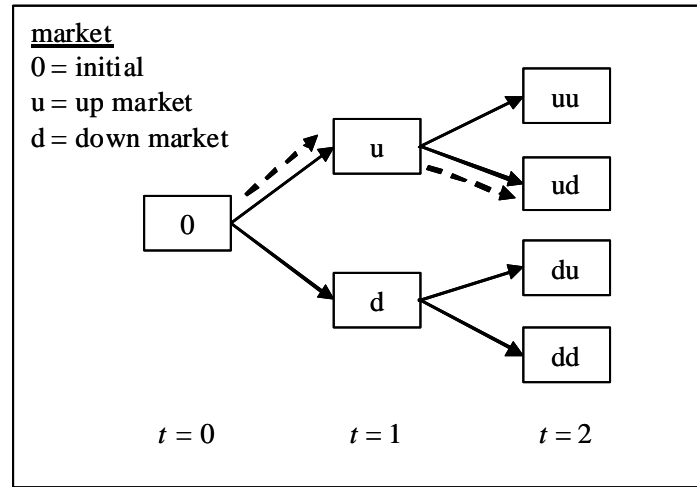


Figure 3. Market tree representation for a 3-period retirement

There are two equally possible market paths in time period 1 and four equally likely market paths in time period 2. One possible retirement path is shown by the dashed arrows in Figure 3 where the retiree experiences one up market and then one down market. For any given retirement of length T , the number of possible market paths is 2^T and the number of consumption nodes is $\sum_{t=0}^T 2^t$.

1. Market Completeness

In our model economy, we follow the example market returns used by Watson (2008). The return on stocks in an up market (R_u) is 1.18 and the return in a down market (R_d) is 0.94. The risk free return on bonds (R_f) is 1.02 regardless of the market condition. The capital market for a three period retirement is represented below as square matrix M where each row represents one of the seven consumption nodes that make up the four possible market paths. Generally, the size of M is $\sum_{t=0}^T 2^t$ by $\sum_{t=0}^T 2^t$. The size of M for our example is 7 by 7.

$$M = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & R_f & R_u & -1 & -1 & 0 & 0 \\ 0 & R_f & R_d & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & R_f & R_u & 0 & 0 \\ 0 & 0 & 0 & R_f & R_d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_f & R_u \\ 0 & 0 & 0 & 0 & 0 & R_f & R_d \end{bmatrix} \begin{matrix} \leftarrow t=0 \\ \leftarrow t=1, s=1 \text{ (u)} \\ \leftarrow t=1, s=2 \text{ (d)} \\ \leftarrow t=2, s=1 \text{ (uu)} \\ \leftarrow t=2, s=2 \text{ (ud)} \\ \leftarrow t=2, s=3 \text{ (du)} \\ \leftarrow t=2, s=4 \text{ (dd)} \end{matrix} \quad (3)$$

where u = up market, d = down market, uu = two consecutive up markets, ud = an up market followed by a down market, etc.

For our example, the capital market conditions are:

$$M = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1.02 & 1.18 & -1 & -1 & 0 & 0 \\ 0 & 1.02 & 0.94 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1.02 & 1.18 & 0 & 0 \\ 0 & 0 & 0 & 1.02 & 0.94 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.02 & 1.18 \\ 0 & 0 & 0 & 0 & 0 & 1.02 & 0.94 \end{bmatrix} \quad (4)$$

An investment plan is represented by vector x where the first element is the initial wealth at the beginning of retirement W_0 . The remaining elements of x correspond to the amounts of remaining wealth to be invested in bonds and stocks for each time t and previous market path s .

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 150.00 \\ 40.00 \\ 60.00 \\ 14.00 \\ 42.00 \\ 24.00 \\ 24.00 \end{bmatrix} \begin{array}{l} \leftarrow t = 0, W_0 \\ \leftarrow t = 1, \text{bond investment} \\ \leftarrow t = 1, \text{stock investment} \\ \leftarrow t = 2, \text{bond investment if previous arc is up} \\ \leftarrow t = 2, \text{stock investment if previous arc is up} \\ \leftarrow t = 2, \text{bond investment if previous arc is down} \\ \leftarrow t = 2, \text{stock investment if previous arc is down} \end{array} \quad (5)$$

Vector c represents the desired consumption plan for each possible time and market path achieved by investing plan x into the capital market M .

$$c = \begin{bmatrix} c_{0,1} \\ c_{1,1} \\ c_{1,2} \\ c_{2,1} \\ c_{2,2} \\ c_{2,3} \\ c_{2,4} \end{bmatrix} = \begin{bmatrix} 50.00 \\ 55.60 \\ 49.20 \\ 63.84 \\ 53.76 \\ 52.80 \\ 47.04 \end{bmatrix} \begin{array}{l} \leftarrow t = 0 \\ \leftarrow t = 1, s = 1 \text{ (u)} \\ \leftarrow t = 1, s = 2 \text{ (d)} \\ \leftarrow t = 2, s = 1 \text{ (uu)} \\ \leftarrow t = 2, s = 2 \text{ (ud)} \\ \leftarrow t = 2, s = 3 \text{ (du)} \\ \leftarrow t = 2, s = 4 \text{ (dd)} \end{array} \quad (6)$$

The relationship between the market M , the investment strategy x , and the desired consumption plan c is:

$$Mx = c \quad (7)$$

A sample 3 period retirement financial plan may resemble:

$$\begin{array}{c}
 M \\
 \begin{bmatrix}
 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
 0 & 1.02 & 1.18 & -1 & -1 & 0 & 0 \\
 0 & 1.02 & 0.94 & 0 & 0 & -1 & -1 \\
 0 & 0 & 0 & 1.02 & 1.18 & 0 & 0 \\
 0 & 0 & 0 & 1.02 & 0.94 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1.02 & 1.18 \\
 0 & 0 & 0 & 0 & 0 & 1.02 & 0.94
 \end{bmatrix}
 \begin{bmatrix}
 150.00 \\
 40.00 \\
 60.00 \\
 14.00 \\
 42.00 \\
 24.00 \\
 24.00
 \end{bmatrix}
 =
 \begin{bmatrix}
 50.00 \\
 55.60 \\
 49.20 \\
 63.84 \\
 53.76 \\
 52.80 \\
 47.04
 \end{bmatrix}
 \end{array} \quad (8)$$

where market conditions M multiplied by investment decisions x determine the consumption plan c for all possible times and market paths of the market.

Our interest lies in determining an optimal consumption plan. We can use relationship (9) to determine an investment strategy given a desired consumption plan c :

$$x = M^{-1}c \quad (9)$$

It is important to note that for any desired consumption plan c there exists a unique investment plan x to support it. Because of this relationship, the market is said to be complete.

2. Market Prices

Shown below is M^{-1} for a 3 period retirement (10). The first row of M^{-1} represents the cost or price vector $p_{t,s}$ of each time period and market path (rounded to four significant digits). That is, in order to receive \$1 in at time $t = 1$ you must invest 0.3268 if the market goes up or 0.6536 if the market goes down. Market performance and prices are negatively correlated. Better market performance (up markets) relates to a cheaper cost of consumption in the future. The relationship is the same for down

markets. The remaining numbers in each column of M^{-1} show the investment strategies required to support desired consumption. A thorough numerical example is given in Sharpe (2007b).

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & \underbrace{\quad\quad}_{t=0} & \underbrace{\quad\quad\quad}_{t=1} & \underbrace{\quad\quad\quad\quad\quad\quad}_{t=2} \\
 s = & - & 1 \text{ (u)} & 2 \text{ (d)} & 1 \text{ (uu)} & 2 \text{ (ud)} & 3 \text{ (du)} & 4 \text{ (dd)}
 \end{array} \\
 M^{-1} = \begin{bmatrix}
 1.0000 & 0.3268 & 0.6536 & 0.1068 & 0.2136 & 0.2136 & 0.4272 \\
 0 & -3.8399 & 4.8203 & -1.2549 & -2.5097 & 1.5752 & 3.1505 \\
 0 & 4.1667 & -4.1667 & 1.3617 & 2.7233 & -1.3617 & -2.7233 \\
 0 & 0 & 0 & -3.8399 & 4.8203 & 0 & 0 \\
 0 & 0 & 0 & 4.1667 & -4.1667 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -3.8399 & 4.8203 \\
 0 & 0 & 0 & 0 & 0 & 4.1667 & -4.1667
 \end{bmatrix} \quad (10)
 \end{array}$$

C. MODEL DEVELOPMENT

In this section, we formulate the problem of maximizing expected utility in retirement as a non-linear program. This program will be referred to as the maximum utility retirement program (MURP).

1. MURP Formulation

A description of MURP follows:

Indices

t time period where $t \in \{-1, 0, 1, \dots, T\}$

s market path where $s \in \{1, 2, \dots, 2^T\}$

Data

W_0 initial wealth

- $c_{-1,1}$ consumption previous to the first year of retirement
- S_t number of market paths at time t , $S_t = 2^t, \forall t \geq 0, S_{-1} = 1$
- $\pi_{t,s}$ probability that market path s will occur at time t , $\pi_{t,s} = 1/S_t$
- $p_{t,s}$ price today of asset at market path s at time t
- d_t habit forming coefficient at time t , typically $d_t \in [0,1], \forall t$
- a_t time discount factor at time t , typically $a_t \in [0,1], \forall t$
- g_t risk aversion power at time t , typically $g_t \in [3,4], \forall t$

Variables

- $c_{t,s}$ consumption at time $t \geq 0$ and market path s
- c consumption vector where $c = [c_{0,1}, c_{1,1}, c_{1,2}, \dots, c_{T,S_T}]'$

Functions

$$U(c) = \sum_{t=0}^T \sum_{s=1}^{S_t} \pi_{t,s} U_{t,s}(c_{t,s}, c_{t,\lceil s/2 \rceil}) \quad \text{present time expected utility of consumption over all time periods and market paths.}$$

Mathematical Formulation of MURP

$$\max U(c) \tag{11}$$

s.t.

$$\sum_{t=T} \sum_{s=S_t} p_{t,s} c_{t,s} = W_0 \tag{12}$$

Equation (11) defines the objective function which is the total expected utility experienced for a retirement of length T . Constraint (12) ensures that all wealth is consumed. We note that it is not necessary to impose the constraint $c_{t,s} \geq 0, \forall t$ because of the form of the utility function.

D. ALGORITHM

1. Solution Technique

For smaller problems, i.e., $T = 0, 1$, or 2 , it is relatively simple to solve for an optimal consumption plan using the method of Lagrange multipliers (Watson, 2008). Compute the first order Karush-Kuhn-Tucker (KKT) necessary and sufficient conditions for constrained optimality:

$$\nabla L(\lambda, c) = \nabla U(c) + \lambda \begin{bmatrix} p_{0,1} \\ p_{1,1} \\ p_{1,2} \\ p_{2,1} \\ p_{2,2} \\ \vdots \\ p_{T,s_T} \end{bmatrix} = 0 \quad (13)$$

The first order KKT conditions for a two period ($T = 1$) retirement problem are:

$$(c_{0,1} - d_0 c_{-1,1})^{-g_0} + \pi_{1,1} a_1 (c_{1,1} - d_1 c_{0,1})^{-g_1} (-d_1) + \pi_{1,2} a_1 (c_{1,2} - d_1 c_{0,1})^{-g_1} (-d_1) = \lambda p_{0,1} \quad (14)$$

$$\pi_{1,1} a_1 (c_{1,1} - d_1 c_{0,1})^{-g_1} = \lambda p_{1,1} \quad (15)$$

$$\pi_{1,2} a_1 (c_{1,2} - d_1 c_{0,1})^{-g_1} = \lambda p_{1,2} \quad (16)$$

$$c_{0,1} + p_{1,1} c_{1,1} + p_{1,2} c_{1,2} = W_0 \quad (17)$$

By substituting for $c_{0,1}$, $c_{1,1}$, and $c_{1,2}$ with known parameters (p_t , d_t , a_t , g_t , W_0), the wealth constraint (17) can be used to solve for λ . Once λ is determined, we can solve for initial consumption $c_{0,1}$. After solving for initial consumption we can easily solve for $c_{1,1}$ and $c_{1,2}$.

For larger problems, i.e., $T > 2$, we can still apply the method of Lagrange multipliers, but the problem becomes too large to realistically compute by hand. Because the size of c , x and the market matrix M grow exponentially as T increases, computing M^{-1} for the prices $p_{t,s}$ and λ becomes very difficult.

Sharpe, Scott and Watson (2008) present a method for computing the prices $p_{t,s}$. Specifically, they show how to compute the two elementary (single period) prices, ψ_{up} and ψ_{down} , using arbitrage pricing techniques and that future market prices can be written as products of single period prices. Therefore, only the number of market ups and downs matters and not the actual order of ups and downs. The prices of any time and market path can be computed using this technique vice having to compute M^1 . The new equations for market prices are:

$$\psi_{up} = \frac{R_f - R_d}{R_f(R_u - R_d)} \quad (18)$$

$$\psi_{down} = \frac{R_u - R_f}{R_f(R_u - R_d)} \quad (19)$$

$$p_{t,s} = \psi_{up}^{\#up \text{ mkt}s} \psi_{down}^{t-\#up \text{ mkt}s} \quad (20)$$

where “#up mkt’s” is the cumulative number of up markets experienced in time periods $0, 1, 2, \dots, t$.

When attempting to solve for λ in a $T = 30$ problem there are as many unknowns as there are possible market paths $\sum_{t=0}^T 2^t$. Watson (2008) derives a reduced budget constraint consisting only of known parameters using algebraic reduction, recursion and the binomial theorem. The rewritten budget constraint is as follows:

$$W_0 = B_0 d_0 c_{-1,1} + \sum_{t=0}^T B_t \left(\frac{a_t}{\lambda B_t} \right)^{1/g_t} \left(\psi_u^{1-1/g_t} \pi_u^{1/g_t} + \psi_d^{1-1/g_t} \pi_d^{1/g_t} \right)^t \quad (21)$$

where

$$B_0 = 1 + \frac{d_1}{R_f^1} + \frac{d_1 d_2}{R_f^2} + \dots + \frac{d_1 d_2 \dots d_T}{R_f^T} \quad (22)$$

$$B_t = 1 + \left(\frac{d_{t+1}}{R_f} \right) B_{t+1} \quad (23)$$

$$B_T = 1 \quad (24)$$

For a large problem, (21) still presents a significant challenge, but because it is made up of known (computable) parameters the solution for λ can be quickly found via a

bisection root finding algorithm. Given λ , we use backward substitution to compute the consumption $c_{t,s}$ for all time periods and market paths.

E. MURP SOLVER AND EXCEL INTERFACE

To develop a planning tool, we implement the MURP solution algorithm in C++ using the integrated development environment Dev-C++ (v.5 Beta) from Bloodshed Software (www.bloodshed.net) that is distributed under the GNU general public license.

A spreadsheet interface is designed in Microsoft Excel and allows input of initial parameters, execution of the MURP solver, and import and subsequent analysis of the results.

When called, the MURP solver solves for optimal retirement consumption paths for all possible times and market paths according to the input parameters. Because it is not sensible to extract all possible paths for larger values of T , a uniformly distributed random sample of N market paths of length T time periods from a population of 2^T total paths is drawn. The statistics of those paths (average consumption, average minimum consumption, average maximum consumption, standard deviation, and market performance) are computed and automatically imported into the Excel interface for analysis. The MURP Excel interface screenshot in Figure 4 shows how graphs can be created showing the range of possible retirement path average consumption and standard deviation. We call this collection of paths ‘retirement clouds.’

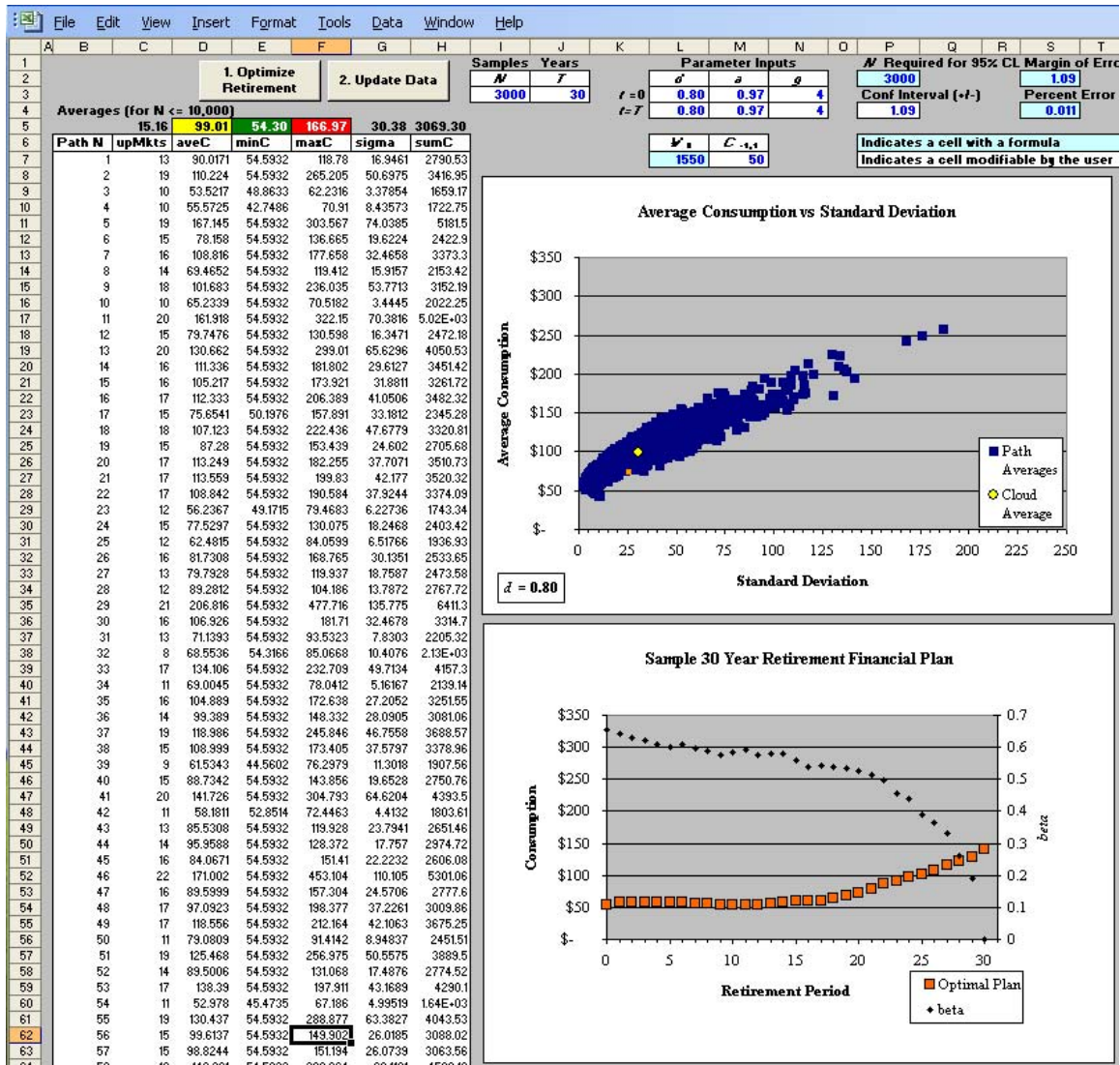


Figure 4. MURP Excel interface screenshot

III. NUMERICAL ANALYSIS

A. EFFECTS OF HABIT FORMATION ON CONSUMPTION

Table 1 shows inputs for a typical retirement scenario with the exception of the habit forming parameter. A random sample of size $N = 3000$ market paths of a total population of 2^T paths was chosen to achieve a 1.1% margin of error or less with a 95% confidence level for average consumption. As defined in Chapters I and II, T is the retirement horizon, d_t , a_t , and g_t are personal preferences, W_0 is initial wealth, and $c_{-1,1}$ is the consumption previous to the first year of retirement. For our analysis, we assume personal preferences remain constant.

N	T	d_t	a_t	g_t	W_0	$c_{-1,1}$
3000	30	See Table 2	0.97^t	4.0	\$1,550,000	\$50,000

Table 1. Scenario parameter inputs

Seven different sample results are presented in Table 2 to show the effects of habit formation on optimal retirement financial plans. *Mkt Perf* is the average number of up markets experienced in the sample of 30 year retirement plans. *Mean C* and *Std Dev* are the average consumption and average standard deviation in the sample. The *95% C.I.* +/- column shows the *Mean C* confidence interval for the sample.

d_t	<i>Mkt Perf</i>	<i>Mean C</i>	<i>Std Dev</i>	<i>95% C.I. +/-</i>
0.00	15.00	103.70	27.64	0.99
0.25	15.00	103.40	28.22	1.01
0.50	14.98	102.06	28.30	1.01
0.75	15.00	99.16	29.41	1.05
0.88	14.99	93.60	28.67	1.03
0.96	14.99	84.99	25.30	0.91
1.00	15.00	75.20	17.94	0.64

Table 2. Scenario MURP results

At $d_t = 0$, habit formation is non-existent and the results are the same as would be for time separable utility. As d_t increases, expected optimal consumption of a retirement financial plan is slowly decreased, but the standard deviation tends to remain steady and does not begin decreasing until $d_t = 0.9$ and greater. As d_t approaches 1.0, the previous year's consumption becomes more important and future spending decreases become intolerable. Average consumption and standard deviation are greatly reduced resulting in a much less erratic retirement consumption path. Gomes and Michaelides (2003) refer to this type of reduction in volatility as consumption 'smoothing.'

Figures 5 through 7 are plots of 'retirement clouds' associated with the results in Table 2. Each data point in the cloud represents the average expected consumption over 30 years of an optimized retirement financial plan. The entire cloud is the range of expected consumption and standard deviation for all possible market paths. Given any combination of market performance the retiree is assured that his/her expected consumption will reside somewhere inside that cloud and will provide the maximum expected utility. The graphs also show the average expected consumption over all paths and is labeled "cloud average."

The figures show that as d_t increases, average consumption and standard deviation decrease as well resulting in a more closely consolidated cloud of optimal retirement plans.

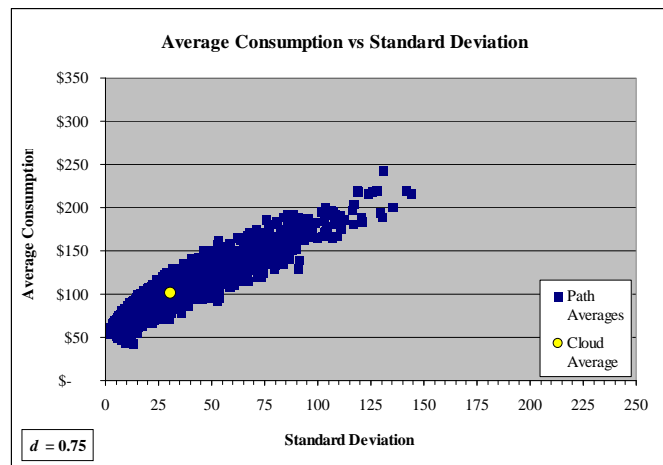


Figure 5. Optimized retirement plot for $d_t = 0.75$

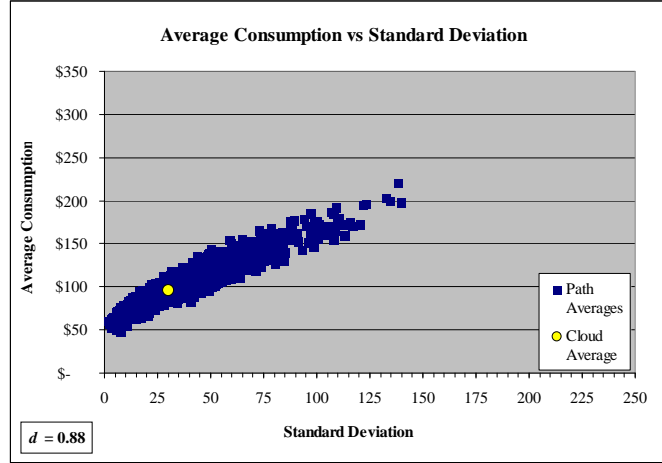


Figure 6. Optimized retirement plot for $d_t = 0.88$

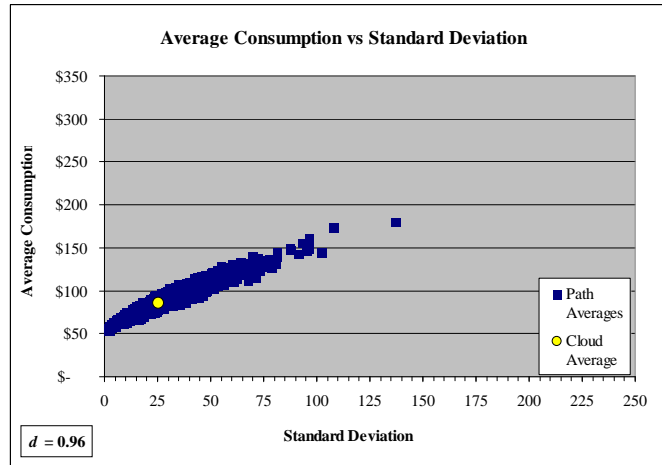


Figure 7. Optimized retirement plot for $d_t = 0.96$

The standard deviation of consumption for a MURP optimization is positively correlated with the expected average consumption. Figure 8 shows that higher levels of volatility are associated with retirement paths with higher numbers of up markets. As shown in Figures 5-7, retirement paths with higher numbers of up markets experience higher levels of consumption. Therefore, volatility is due to more consumption vice less.

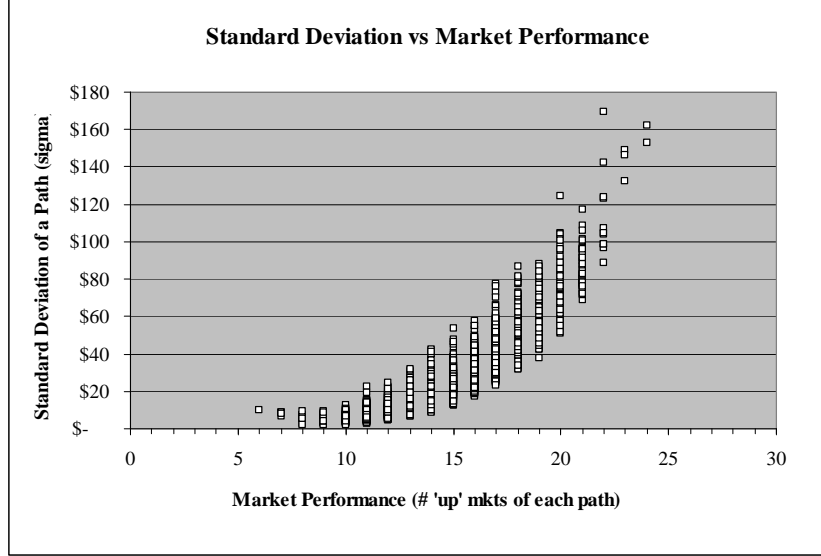


Figure 8. Standard Deviation vs. Market Performance

Retirement paths with poor market performance have low standard deviations because the habit formation utility function protects the retiree from large decreases in consumption during periods of down markets. Specifically, the maximum decrease in consumption from one period to the next is $(1 - d_t)c_{t-1, \lceil s/2 \rceil}$.

B. MURP OPTIMIZED RETIREMENT CASE STUDY

Consider an individual named Joe who is entering retirement with \$1,550,000. Joe wishes to choose a retirement financial plan that maximizes his utility throughout his 30 year retirement and will be investing and consuming as per our retiree, capital market, and retirement scenario in Chapter I. Joe's habit formation, risk aversion, and time discount factor are 0.80, 4.0 and 0.97, respectively. Again, we assume personal preferences remain constant throughout retirement. Table 3 shows all of the parameters for Joe's scenario.

N	T	d_t	a_t	g_t	W_0	$c_{-1, 1}$
3000	30	0.80	0.97^t	4.0	\$1,550,000	\$50,000

Table 3. Retirement scenario inputs

We choose $N = 3000$ in order to reduce the sample margin of error for average consumption to less than 1.5%. Joe is presented with his optimized retirement cloud obtained by the MURP Excel interface shown in Figure 9. Given all possible combinations of market ups and downs (market paths) Joe is assured that his plan resides within the boundaries of the cloud.

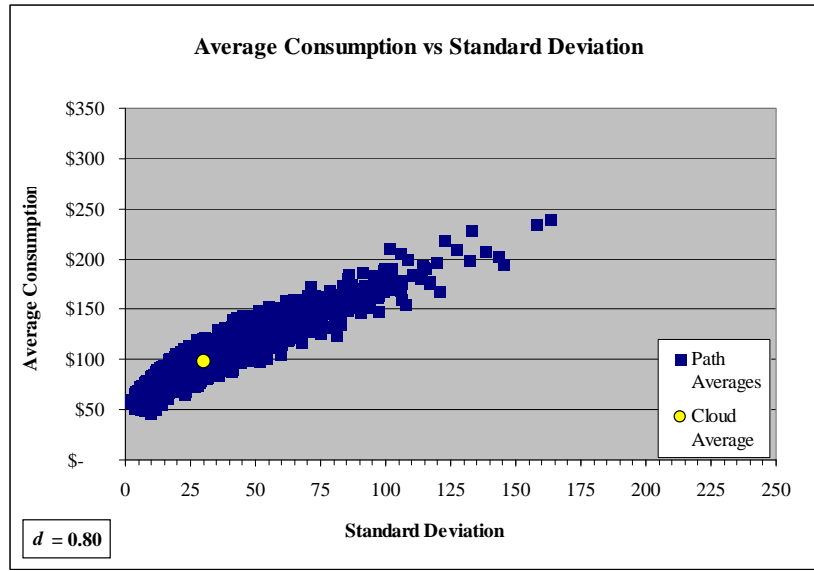


Figure 9. Sample Optimized Retirement Cloud

While explaining Figure 9 to Joe, it is important to note that standard deviation of consumption for a MURP optimization is positively correlated with the expected average consumption. A higher level of habit formation protects retirees from large negative variations in consumption but allows for significant increases when supported by favorable (up) markets. Therefore, for the MURP optimization model, Joe can associate higher standard deviation with more expected consumption.

Figure 10 is a plot of one data point from the area above and to the right of the “cloud average” data point in Figure 9. This is a period to period example of an optimal consumption and investment plan given the market path in Table 4. The cumulative market performance over time is shown by “Mkt Ups” and increases every year there is an up market. Down markets do not affect the total.

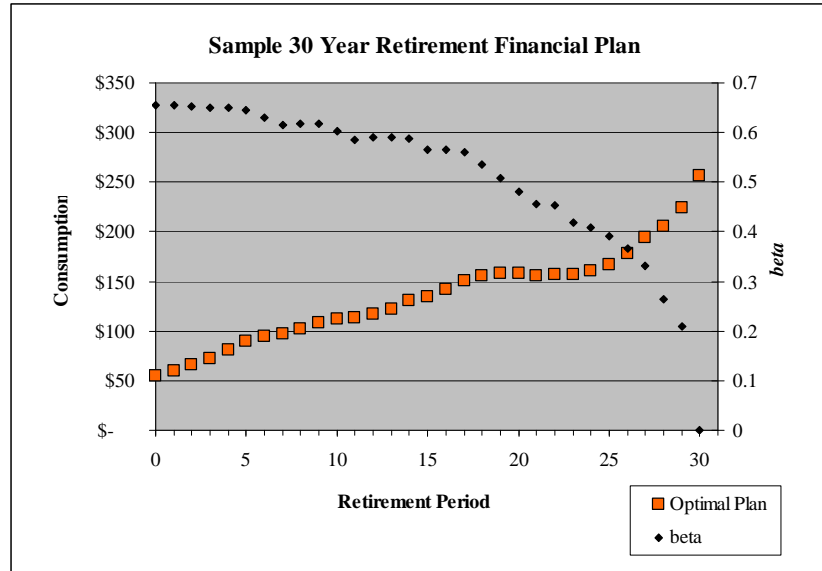


Figure 10. Sample Optimized Retirement Financial Plan 1

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Mkt Ups	0	1	2	3	4	5	5	5	6	7	7	7	8	9	10	10	11	12	12	12	12	12	13	13	14	15	16	17	17	18	19

Table 4. Market Path for Sample Plan 1

Consumption is plotted against the left vertical axis and the associated investment plan β (beta) required to support this spending plan is plotted on the right vertical axis. The parameter β is the proportion of remaining wealth to be invested into the risky stock market, and $1 - \beta$ is the proportion to invest in risk-free bonds. As per this chart, in the early years of retirement greater than 65% of Joe's portfolio is invested in stocks because of their expected higher return, but the stock portfolio percentage decreases as Joe progresses through retirement. In the 30th year, $\beta = 0$ because this is the year that all remaining wealth is withdrawn and consumed. Regardless of the market path, β decreases to zero for every MURP optimized retirement.

For this particular path, Joe does quite well. The market goes up 19 times, which is above the average amount of 15, and Joe's consumption skyrockets from \$50,000 to over \$250,000.

It is easy to see that up markets produce increases in consumption. When markets decrease, consumption increases become minimal with possible small decreases during

multiple years of down markets. As mentioned, in every MURP optimization β is continually decreasing through retirement, but up markets allow for β to decrease more slowly while down markets tend to accelerate the reduction.

Figure 11 shows the details of another path from the cloud in Figure 9. Plan 2 was taken from the lower left region of the above cloud where there is lower expected consumption and standard deviation. As shown in Table 5, this path had only 12 up markets during the 30-year retirement.

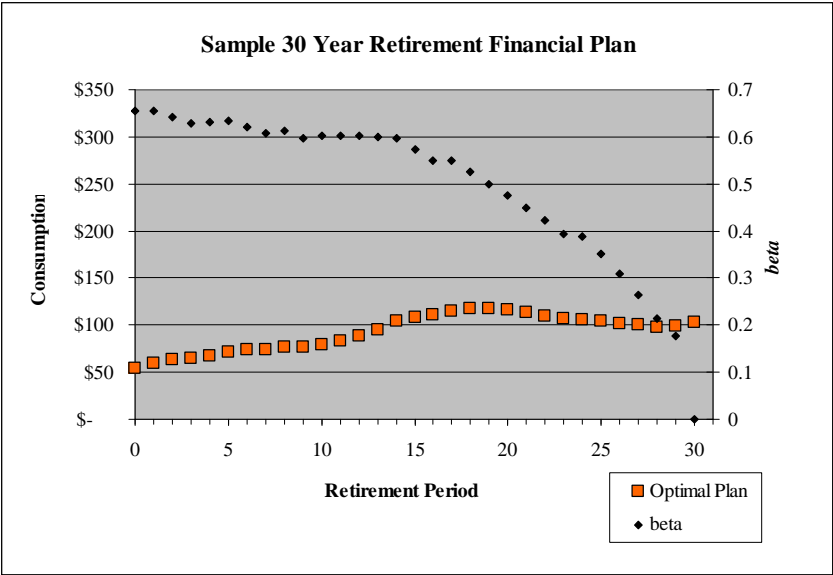


Figure 11. Sample Optimized Retirement Financial Plan 2

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
Mkt Ups	0	1	1	1	2	3	3	3	4	4	5	6	7	8	9	9	9	10	10	10	10	10	10	10	10	11	11	11	11	11	12	12

Table 5. Market Path for Sample Plan 2

During an 11 year period, $t = 18-28$, there is only one up market and Joe’s consumption suffers when compared to consumption in Plan 1. The good thing is that the MURP optimized retirement plan is dynamic and responds to different market environments. Joe’s wealth is protected and keeps him safely out of the work force throughout retirement.

C. MURP OPTIMIZED VS THE 4% RULE

As described in Scott, Sharpe and Watson (2009), for a 30-year retirement horizon, many financial planners using the 4% Rule actually suggest a drawdown rate between 4% and 5%. Here we use the midpoint value of 4.5% which gives us a drawdown rate of \$69,750 per year. After withdrawing the annual amount of \$69,750, the portfolio must be rebalanced to ensure a 60%-40% mix of risky stocks and bonds which corresponds to the risk level and return of the risky asset in our two asset capital market described in Chapter II.

The 4% Rule is considered to be a static financial strategy because annual consumption is based solely off initial wealth and without regard to the market conditions. When the market is weak, MURP consumption can decrease to protect future spending and can increase if the market is strong. The 4% Rule generally provides more consumption consistency than a MURP optimized retirement plan, but, because it is static, it fails to take advantage of the favorable markets, and fails to protect investors during unfavorable markets. As described in Scott, Sharpe, & Watson (2009), a surplus is created by the 4% Rule when the investment portfolio returns exceed desired consumption, and a shortage is created when returns fail to support desired consumption. In either case, the portfolio is said to be inefficient. The MURP optimized retirement is dynamic in that it modifies consumption based on fluctuations in the market and efficient because it does not build up wealth surpluses and shortages.

Where the 4% Rule struggles most is in market paths that experience a below average number of up markets (less than 15); especially if the down markets are concentrated in the first half of retirement. Figure 12 is a comparison of a MURP optimized retirement path and the retirement experienced under the 4% Rule given the same market path in Table 6. Here, the MURP optimized retirement plan protects the retiree's wealth by reducing consumption during the extended down market years and then safely takes advantage of the up markets and increases consumption at the end of retirement. The average consumption for the MURP retirement is \$63,780. The average consumption for the 4% Rule should be \$69,750 but was only \$56,750 because wealth

was completely exhausted by the 26th year. So, under the 4% Rule, the retiree would actually be forced to go back to work for his/her last few years of retirement.

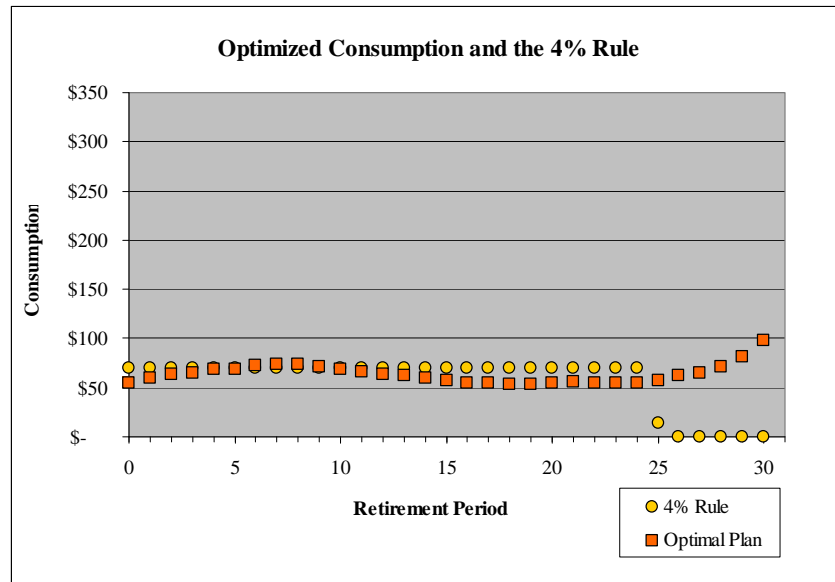


Figure 12. Sample Optimized Consumption Plan 3 and the 4% Rule

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Mkt Ups	0	1	1	1	2	2	3	3	3	3	3	3	4	4	4	4	5	5	6	7	7	7	7	8	9	10	10	11	12	13	

Table 6. Market Path for Sample Plan 3

Another challenge for the 4% Rule is that for retirements with above average market performance where the retiree builds excess wealth while consumption remains consistent. This case is shown in Figure 13 and Table 7 where the market was very strong with a total of 20 up years. For this market path the 4% Rule built a huge surplus of wealth totaling \$8,992,010 by the end of retirement. If indeed a retiree wants only to spend \$69,750 each year as planned for in the 4% Rule, the static investment strategy exposes him to much more risk than necessary. But even if the retiree prefers more spending to less, the 4% Rule's static consumption strategy denies him/her of possible increases. Our retiree prefers more consumption to less and the MURP optimized retirement consumption increased as allowable resulting in an average of \$129,030.

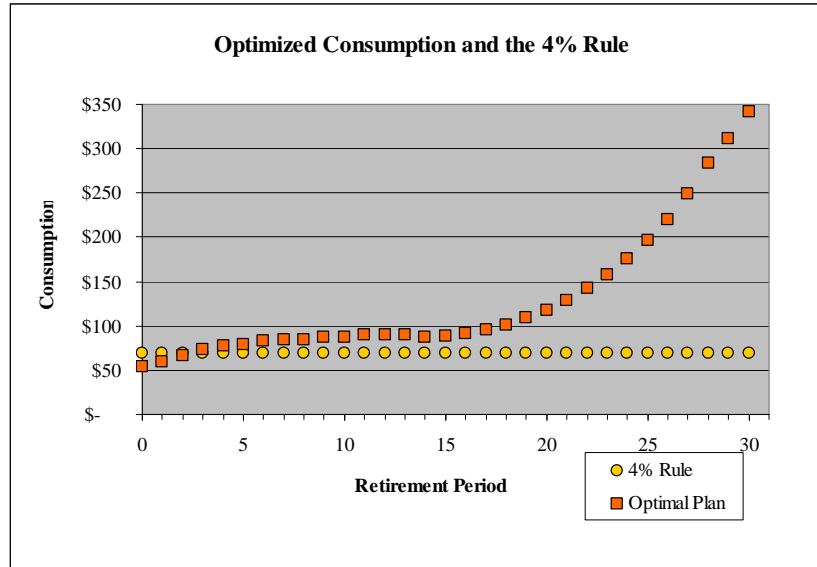


Figure 13. Sample Optimized Consumption Plan 4 and the 4% Rule

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Mkt Ups	0	1	2	3	3	3	4	4	4	5	5	6	6	6	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	20	20

Table 7. Market Path for Sample Plan 4

Here, the 4% Rule does not fail to carry the retiree through retirement without going bankrupt, but it is very inefficient in that it does not afford the retiree with more consumption, it exposes the retiree to too much risk for the static spending plan, and it creates large amounts of excess wealth.

The 4% Rule is also very market path dependent. It can survive a poor economy if there are sufficient numbers of up markets in the beginning of retirement but it can just as easily fail if there too few in the early years. For this scenario, the 4% Rule can remain solvent with as little as 7 total up markets and must have all of those in the first half of retirement. On the other hand, the 4% Rule can still fail even with up to 21 years of up markets if the first up market does not occur until year $t = 10$. But these are extreme cases and the probability of our market experiencing either one is less than 0.27%. Since both up and down markets are equally likely, our market averages 15 ups in a 30-year retirement.

Figure 14 and Table 8 show a retirement path with a below average of 12 total up markets. The first up market is experienced in year $t = 2$, and 6 of the 12 up markets are experienced during the first 11 years.

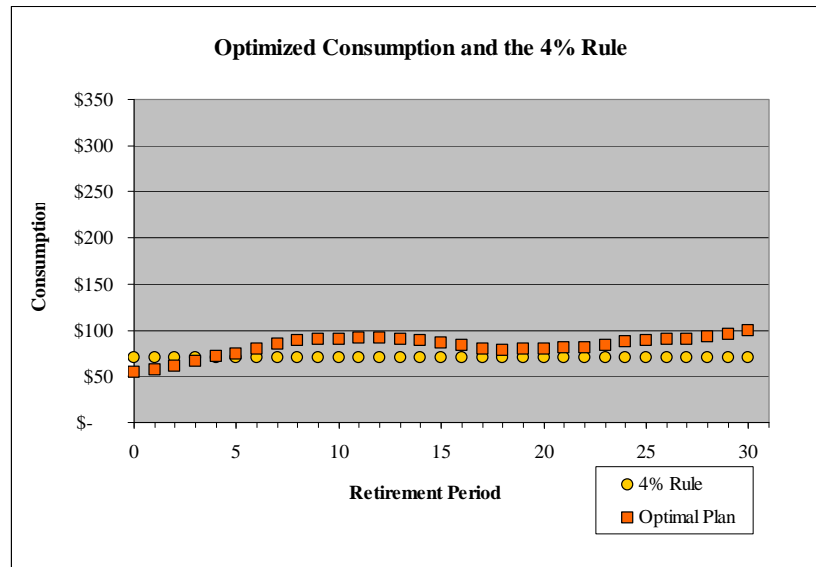


Figure 14. Sample Optimized Consumption Plan 5 and the 4% Rule

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Mkt Ups	0	0	1	2	3	3	4	5	5	5	5	6	6	6	6	6	6	6	7	8	8	9	9	10	11	11	11	11	12	12	12

Table 8. Market Path for Sample Plan 5

Both MURP and the 4% Rule supported the retiree to the end of retirement with average spending for this path at \$82,520 for MURP and \$69,750 for the 4% Rule. Even with a below average market, the 4% Rule manages to accumulate a surplus wealth of \$663,300. But what if the market sequence was different?

Figure 15 and Table 9 show just how profound of an effect market patterns can have on retirement consumption. This path also has 12 up markets, but there is a distinct difference from before in that it only has 4 up markets in the first half of retirement with the first one not occurring until year 4. Here, the 4% Rule fails at year 21 and average consumption is reduced to \$47,220 because there is zero wealth to consume for the last 10 years of retirement. Average consumption for the MURP optimized path is reduced

by \$21,820 from the Figure 14 path to \$60,700. Such a large reduction could prove to be a difficult adjustment for any retiree to make, but it would still be better than the alternative of bankruptcy given by 4% Rule in this case.

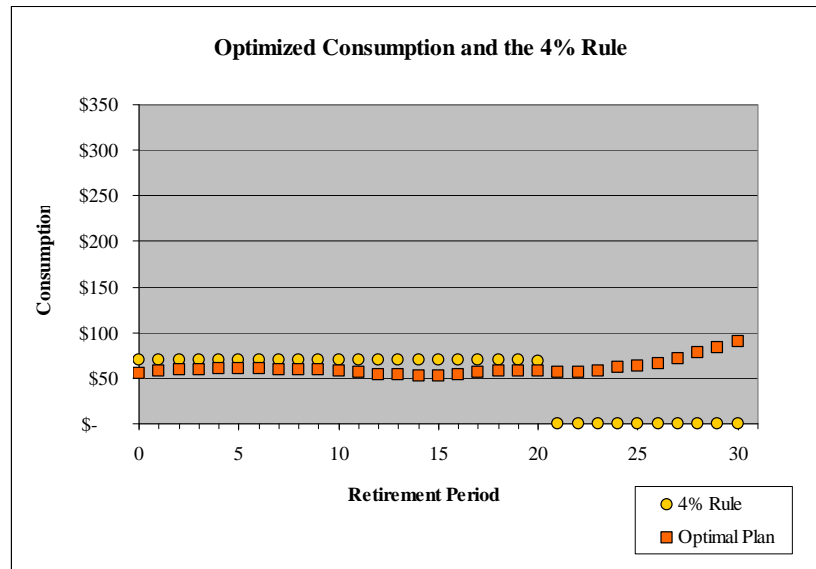


Figure 15. Sample Optimized Consumption Plan 6 and the 4% Rule

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Mkt Ups	0	0	0	0	1	1	1	1	2	2	2	2	2	3	3	4	5	6	6	6	6	6	7	8	9	9	10	11	12	12	12

Table 9. Market Path for Sample Plan 6

A final example given in Figure 16 and Table 10 shows consumption for a market path with 15 up markets, but because the market performed so poorly at the onset of retirement the 4% Rule was unable to build up sufficient wealth and begins to fail in year 24. Since the MURP optimization accounts for uncertainty in markets, it allows for a retiree to efficiently and successfully navigate through any market path without fear of a shortfall, and instead of running out of money in the 24th year of retirement the retiree enjoys a nice upswing in consumption during the last 6 years. The MURP optimized and 4% Rule average consumption rates for this example are \$74,220 and \$55,270, respectively.

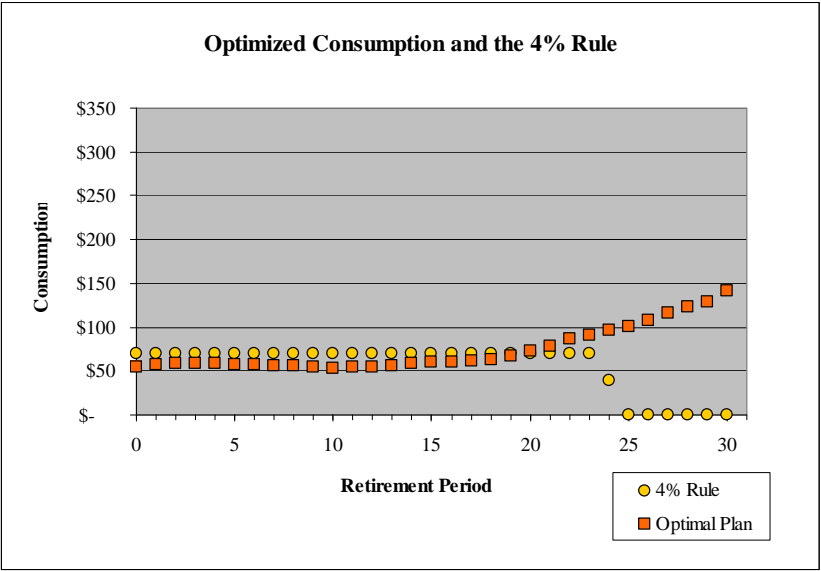


Figure 16. Sample Optimized Consumption Plan 7 and the 4% Rule

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Mkt Ups	0	0	0	0	0	0	1	1	1	1	2	3	3	4	5	5	5	6	7	8	9	10	11	11	12	12	13	14	14	14	15

Table 10. Market Path for Sample Plan 7

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IV. CONCLUSION AND RECOMMENDATIONS

We have implemented an algorithm for solving and analyzing the nonlinear maximum utility retirement program (MURP) and demonstrated the effects of habit formation on optimal retirement consumption and investment plans. We also showed how MURP can solve for optimal retirement financial plans for our two-asset binomial model economy and 30-year retirement scenario.

The MURP Excel interface we developed enables any habit forming individual investor to solve for an optimal retirement financial plan for every possible combination (path) of market performances. Using the MURP Excel interface, we can also graphically demonstrate representative samples of these optimal plans in collections called ‘retirement clouds.’ These clouds exhibit the range of a retiree’s expected consumption for all optimal retirement financial plans. A tool such as this could prove to be very useful for individuals managing their own retirement accounts with limited resources. As such, this method could be employed as an alternative to using rules of thumb such as the 4% Rule which can be wasteful and many times unreliable. With MURP optimization, retirees could be afforded with a more efficient, safe, and personalized method for developing optimal utility maximizing financial strategies.

This study limited the habit formation coefficient to values between 0.0 and 1.0. Current studies and surveys are showing the possibility of the habit formation coefficient to be greater than 1.0 for some individuals and possibly even negative. The MURP optimization can be reformulated or expanded to allow for greater ranges of habit formation. Our scenario also assumed that individuals know their consumption preferences (risk aversion, time discount, and habit formation) upon entering retirement. In reality these preferences are difficult to quantify. MURP retirement clouds can be used help determine someone’s personal preferences by allowing investors to choose between clouds generated with varying parameters. Finally, our model economy was limited to only two states and two securities. Including additional states and investments opportunities will increase the usefulness of the MURP optimization tool.

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